



Estimating satisfiability

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ABSTRACT

The problem of estimating the proportion of satisfiable instances of a given CSP (constraint satisfaction problem) can be tackled through weighting. It consists in putting onto each solution a non-negative real value based on its neighborhood in a way that the total weight is at least 1 for each satisfiable instance. We define in this paper a general weighting scheme for the estimation of satisfiability of general CSPs. First we give some sufficient conditions for a weighting system to be correct. Then we show that this scheme allows for an improvement on the upper bound on the existence of non-trivial cores in 3-SAT obtained by Maneva and Sinclair (2008) [17] to 4.419. Another more common way of estimating satisfiability is ordering. This consists in putting a total order on the domain, which induces an orientation between neighboring solutions in a way that prevents circuits from appearing, and then counting only minimal elements. We compare ordering and weighting under various conditions.

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1. Introduction

Constraint satisfaction problems cover a large variety of problems that arise in many areas of combinatorial optimization. They are central in complexity theory because they are *NP*-complete and also because one particular case – satisfiability of Boolean formulas – was the first problem to be identified in this class. In general, they consist in defining constraints on a set of *variables* taking their *values* in a given finite domain. *Constraints* specify which combinations of values assigned to subsets of variables are allowed (or dually are forbidden). A *solution* is a valuation (i.e. the assignment of a value to each variable) that does not violate any constraint. The satisfiability problem is the following: given an instance, decide the existence of a solution for it.

Besides the design of algorithms for solving these problems, the research of structural properties for these problems has attracted much attention in the recent years. In particular, the empirical evidence of the existence of a threshold (rigorously established in some particular cases) in the satisfiability of some classes of CSPs has opened a field of research: attempts are made to rigorously establish the existence and the location of this threshold. This involves estimating the proportion of satisfiable instances in a given set of instances. The *NP*-completeness of these problems in general makes it difficult to determine whether a given instance is satisfiable; that may explain why direct counting of satisfiable instances is currently unfeasible. However, precisely because these problems are in *NP*, it is easy to determine whether some instance is satisfied by a given valuation and then to count the formulas satisfied by this valuation. Thus counting couples (formulas, solutions) is only accessible starting from a solution; moreover, given a solution, it is not complicated to investigate also its immediate neighborhood. But even at a distance of 2, i.e. with neighbors of neighbors, calculations become quite complicated (see [15]). This fact imposes a strong restriction on the design of both estimation techniques studied: they can only make use of local information. We shall refer to this as *the locality condition*.

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Using one of the most popular techniques in the probabilistic method (cf. [5]), namely the first moment method, it is possible to bound from above the probability of satisfiability. The implementation of the first moment method makes use of Markov's inequality; one needs to define a non-negative random variable X that must be at least 1 for a satisfiable formula (we call that a *correct* random variable). Ideally, X should be as small as possible; in other words, it should be 0 for unsatisfiable instances and as close to 1 as possible for satisfiable ones (if X is 1 for every satisfiable instance and 0 for every unsatisfiable instance then we get the exact probability of satisfiability). The most straightforward candidate for X is simply the number of solutions. In order to compute the expected number of solutions for a random formula, it suffices to count for each valuation the number of instances that are satisfied by it and then to sum up over all valuations (by linearity of expectation). But since the number of solutions is generally too large, the method over-estimates the proportion of satisfiable instances.

Many techniques have been developed to overcome this difficulty in various types of CSPs: Satisfiability of CNF formulas [13,10,15,11,12,7,14,9,8]; 3-Coloring of graphs [1]; Binary CSPs [2,3] Most of these methods share a common point: they count minimal elements under some partial order over solutions. We will refer to this method as *solution selection through a partial ordering* or for short *ordering*. Due to the locality condition, the partial order must be locally computable (i.e. must depend only on the immediate neighbors of the considered solution). Two solutions of some instance are neighbors if they disagree only on the value taken by one variable. Both solutions may be ordered using a predetermined order on the values for this particular variable in this particular instance. Finally we count only those solutions having minimal values for all their variables with respect to their neighbors.

Recently, Maneva et al. [16] introduced a novel approach for the Boolean satisfiability problem consisting in weighting trivaluations (over $\{0, 1, *\}$) and solutions depending on their neighborhood. While not originally intended to estimate the proportion of satisfiable instances (but rather to analyze some properties of Belief Propagation algorithms), it was nevertheless specifically used by Maneva and Sinclair [17] to estimate the probability of existence of non-trivial cores in random 3-SAT instances. The existence of non-trivial cores contains important information on the structure of the space of solutions; moreover it is related to the clustering that has been proved to exist in k -SAT for $k \geq 9$ Achlioptas and Ricci-Tersenghi [4]. Maneva and Sinclair [17] show that in the 3-SAT instances, non-trivial cores do not exist for ratios of clauses to variables greater than 4.453. To do so they use *valid trivaluations* (i.e. satisfying some properties related to Boolean satisfiability) and weight them according to their values and their neighborhood.

2. Overview of results

Our first result consists in giving some sufficient conditions to make a weighting scheme correct for the estimation of satisfiability on general CSPs ([Theorem 9](#), *Weight Conservation Theorem*). Then we propose a general weighting scheme obeying these conditions ([Theorem 15](#)). This scheme is based on:

1. a *weighting seed* that expresses the relative importance of each value with respect to a variable and an instance; the seed is such that if all valuations were solutions, then their total weight would be exactly 1;
2. a *dispatching function* expressing how the weights of forbidden valuations are dispatched among solutions to insure that counting weighted solutions will yield at least 1 for any satisfiable instance.

We will refer to this method as a *solution weighting* or for short *weighting*.

In [Theorem 21](#), we show that the estimation of satisfiability used by Maneva and Sinclair [17] can be improved upon by using a weighting scheme based on a 3-valued CSP and obeying the conditions of our *Weight Conservation Theorem* (which shows that these conditions are somehow relevant). Thanks to this weighting system, we improve on the upper bound on the existence of non-trivial cores to 4.419. We completely reuse the proof of Maneva and Sinclair [17] for our new weighting system, showing that the improvement on the value of the bound is indeed due to a better weighting system.

Till now the only way to compare ordering and weighting was to compute the estimations of satisfiability obtained by each of them on a certain set of instances and to choose the best one. We give some results comparing these two ways of estimating satisfiability in the following cases:

- weighting and ordering can be instance dependent when such syntactic properties as the number of occurrences of variables and values etc. can guide the design of weighting functions and orderings. We show that in the general case where the weighting function is instance dependent and when the weighting is *homogeneous* (i.e. when weighting seeds and dispatching functions are equal), weighting cannot be better than a well chosen instance dependent ordering ([Theorem 32](#));
- in the case where ordering and weighting are instance independent (which is the case of problems where the values are indistinguishable like graph coloring for example) and in the case of sets of instances closed under value renaming (which is the case of almost all sets of instances considered in the literature), we show that weighting and ordering are equivalent on average ([Theorem 38](#)).

3. Framework

A CSP (Constraint Satisfaction Problem) is a triple $F = \langle X, D, C \rangle$ where X is a set of *variables* taking their values in the same finite *domain* D of values, and C is a set of *constraints*. A *constraint* is a couple $\langle \mathbf{x}, R \rangle$ where $\mathbf{x} \in X^k$ and $R \subseteq D^k$ for some integer k . R is interpreted as the tuples of allowed values. A *valuation* is a vector $v \in D^X$; access to coordinate $x \in X$ of v will

be denoted as $v(x)$. It satisfies some constraint $\langle (x_1, x_2, \dots, x_k), R \rangle$ iff $(v(x_1), v(x_2), \dots, v(x_k)) \in R$. A valuation is said to be a *solution* of a CSP instance iff it satisfies all of its constraints.

We consider some sets \mathcal{F} of CSP instances sharing the same set X of variables and the same domain D . In the rest of the paper $n = |X|$ denotes the number of variables, $d = |D|$ the size of the domain. Given a CSP instance F , let $\mathcal{S}(F)$ denote the set of its solutions.

We are interested in the neighborhood of valuations. Given a valuation v and $a \in D$, we define $v_{x \leftarrow a}$ as the valuation obtained from v by changing the value of x to a (including the case when already $a = v(x)$). Given a variable x , two solutions are called *x-adjacent* if they agree on all variables but x : in other words σ and τ are *x-adjacent* iff $\tau = \sigma_{x \leftarrow \tau(x)}$. Note that for each variable x , *x-adjacency* is an equivalence relation on solutions. Bringing together the *x-adjacency* relations with respect to every variable and removing the loops (σ, σ) we get a non-oriented graph on $\mathcal{S}(F)$ that we call a *solutions network*. Let $N_F(\sigma, x)$ denote the equivalence class of σ under *x-adjacency* (i.e. the neighborhood of σ for variable x); note that $N_F(\sigma, x)$ is a clique for *x-adjacency*. Such a clique will play a central role in our weighting system. We are also interested in the different values that x takes in this equivalence class, so we define $A_F(\sigma, x) = \{\tau(x)\}_{\tau \in N_F(\sigma, x)}$. For example in Fig. 1, solutions ab and aa are *y-adjacent*, $N_F(ab, y) = \{ab, aa\}$ and $A_F(ab, y) = \{b, a\}$.

Most of the results in this paper apply to any set of solutions, regardless of which CSP instance has generated them. The sole solutions network can be thought of as the input of the problem. However it should be borne in mind that weightings and orderings cannot be defined using the global knowledge of the whole set of solutions, because of the locality condition: one can only count instances having a given solution, and for each instance the solutions that are neighbors of this solution (rather than all solutions of a given instance). A convenient way to visualize this limitation is to imagine a network of processors (a processor representing a solution) where each processor has knowledge of its neighbors only and must compute from this knowledge its own weight or determine the orientation with respect to its neighbors.

4. Partial ordering of solutions

Given a CSP instance F , various partial orders $<_F$ can be defined on the set of solutions such that for every two adjacent solutions σ and τ of F , we have either $\sigma <_F \tau$ or $\tau <_F \sigma$. The aim of the partial order here is to provide a measure on the solutions network through the number of its minimal elements. Let $\mathcal{M}_{<_F}(F)$ be the set of minimal solutions of F with respect to the order $<_F$.

In the solutions network of F , a partial order $<_F$ can be seen as a circuit-free orientation of the edges of the graph such that an edge goes from τ to σ iff $\sigma <_F \tau$; then minimal elements are vertices with no outgoing edges. In general one seeks partial orderings that have the least number of minimal elements; however the choice is limited because orderings must be chosen according to local criteria only.

Construction of an ordering.

Definition 1. Given a variable $x \in X$, a total strict order $<_{F,x}$ on D gives an orientation between neighboring solutions: $\sigma <_{F,x} \tau$ iff σ and τ are *x-adjacent* and $\sigma(x) <_{F,x} \tau(x)$. Note that $<_{F,x}$ is a partial strict order on the set of solutions, but a total strict order in each clique $N_F(\sigma, x)$.

We can bring all partial orders $<_{F,x}$ together on the set of solutions, as follows: if σ and τ are *x-adjacent* and different, then $\sigma <_F \tau$ iff $\sigma <_{F,x} \tau$. This is possible because two different solutions σ and τ cannot be both *x-adjacent* and *y-adjacent* for two different variables x and y . We say that $<_F$ is the orientation on $\mathcal{S}(F)$ induced by the set $\{(x, <_{F,x})\}_{x \in X}$.

Lemma 2. If $<_F$ is the orientation on $\mathcal{S}(F)$ induced by a set $\{(x, <_{F,x})\}_{x \in X}$, then $<_F$ is circuit-free.

Proof. Suppose on the contrary that there exists a circuit $\sigma_1 <_F \dots <_F \sigma_l <_F \sigma_1$ for some $l \geq 2$. Let us consider the variable x such that $\sigma_1 <_{F,x} \sigma_2$. For any $i \leq l$, either $\sigma_i(x) = \sigma_{i+1}(x)$ (if σ_i and σ_{i+1} are not *x-adjacent*) or $\sigma_i(x) <_{F,x} \sigma_{i+1}(x)$ (if σ_i and σ_{i+1} are *x-adjacent*). Thus $\sigma_1(x) <_{F,x} \sigma_2(x)$ and $\sigma_2(x) \leq_{F,x} \sigma_3(x) \leq_{F,x} \dots \leq_{F,x} \sigma_l(x) \leq_{F,x} \sigma_1(x)$: a contradiction. \square

Corollary 3. The transitive closure of $<_F$ is a strict order relation.

Instance dependent or not.

- *Instance dependent ordering.* In this case, we put for each variable $x \in X$ and each CSP instance F a total order $<_{F,x}$ onto the domain D of possible values. As mentioned above, we (partially) order solutions as follows: let $\sigma \in \mathcal{S}(F)$ and $\tau \in N_F(\sigma, x)$; we have $\sigma <_F \tau$ if and only if $\sigma(x) <_{F,x} \tau(x)$. The motivation for the instance dependent ordering is that some syntactic properties of the CSP instance F can be exploited to define a suitable order for that instance.
- *Instance independent ordering.* This is a particular case of the above ordering, when the total order $<_x$ on D does not depend on F . For some problems, no preferred order can be defined given some instance. This happens in particular when values are indistinguishable because of the symmetry of the problem (e.g. colors in graph coloring).

Examples of orientations. We first give an example (Fig. 1(a)) of an orientation which is not circuit-free, even though it was built from the following local orderings on each individual clique:

- in cliques $\{a, b\}$ for variables x and y , we have $b < a$;
- in cliques $\{a, b, c\}$ for variables x and y , we have $a < c < b$.

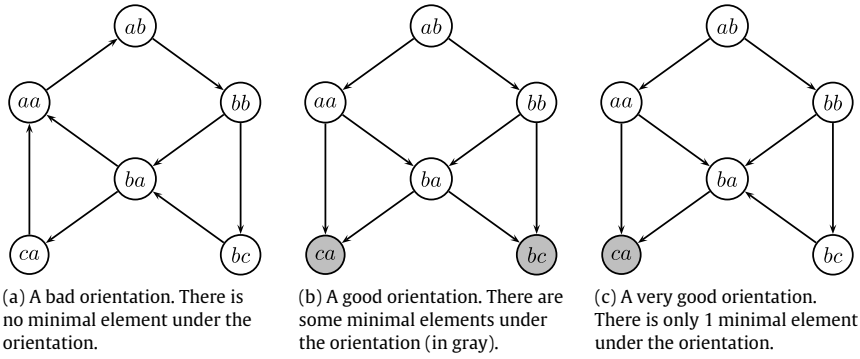


Fig. 1. Some basic examples of orientations. We consider a network of 6 solutions over the domain $D = \{a, b, c\}$ for variables $X = \{x, y\}$; by shortcut ab we mean that variable x takes value a and variable y takes value b . $\{aa, ba, ca\}$ is a clique for variable x .

The problem comes from the fact that a and b are ordered differently in clique $\{a, b, c\}$ and its sub-clique $\{a, b\}$, which led us to consider only orientations built in the following way: we choose for each variable x a total order $<_x$ on the domain D and use it for each sub-clique of D . This is what Boufkhad and Hugel [8] call a *uniform* orientation. The example in Fig. 1(b) was obtained by the following orders: $c <_x b <_x a$ and $c <_y a <_y b$. This orientation is circuit-free and has two minimal elements. Now among good orientations, the less minimal elements they have, the better they are; Fig. 1(c), which was obtained by the following orders: $c <_x b <_x a$ and $a <_y c <_y b$, gives an example of an orientation with just one minimal element.

5. Weighting of solutions

First we define a weighting system for all valuations (solutions or not) which sums up to 1. Then we give sufficient conditions on a weighting system on solutions only, such that a transfer between this weighting system and the previous one may be possible. Doing this we establish a general framework for putting weights onto solutions, and use it to derive two particular weighting systems: the first one addresses general CSPs and the second one is built to improve on the weighting system introduced by Maneva et al. [16], Maneva and Sinclair [17], Ardila and Maneva [6]. The purpose of such a transfer is to estimate the global weight in the weighting system on solutions by means of the global weight of the weighting system on all valuations (which is easier to compute).

5.1. Weighting seeds

Definition 4. For a CSP $F = \langle X, D, C \rangle$ a *weighting seed* is a function $s_F : X \times D \rightarrow \mathbf{R}^+$. We say that s_F is *unitary* iff $\forall x \in X, \sum_{a \in D} s_F(x, a) = 1$.

Now we define the *unladen weight* of any valuation v (solution or not) with respect to some weighting seed s_F as:

$$U_F(v) = \prod_{x \in X} s_F(x, v(x)). \quad (1)$$

As for the *actual weight* of a solution, we want to take into account the neighborhood of the solution, so we put the weight $w_F(\sigma, x)$ on each variable x of solution σ . We will see later how to build w_F from s_F .

The actual weight of a solution is:

$$W_F(\sigma) = \prod_{x \in X} w_F(\sigma, x). \quad (2)$$

By extension, the weight of a set S of solutions is:

$$W_F(S) = \sum_{\sigma \in S} W_F(\sigma). \quad (3)$$

Lemma 5. If the weighting seed s_F is unitary, then the total unladen weight of all valuations is 1: $\sum_{v \in D^X} U_F(v) = 1$.

Proof.

$$\begin{aligned} \sum_{v \in D^X} U_F(v) &= \sum_{v \in D^X} \prod_{x \in X} s_F(x, v(x)) \\ &= \prod_{x \in X} \sum_{a \in D} s_F(x, a) \end{aligned}$$

$$\begin{aligned}
&= \prod_{x \in X} 1 \\
&= 1. \quad \square
\end{aligned}$$

This weight U_F is indeed simple to handle. The purpose is now to connect it with W_F . Just as we defined weights W_F of solutions in a product form variable per variable, so shall we build our transfer system.

5.2. Decomposers

Definition 6. We say that w_F is *decomposed* by a family $(\delta_{F,\sigma,x,a})$ of non-negative real numbers iff for every solution σ of F and every variable x , $w_F(\sigma, x) = \sum_{a \in D} \delta_{F,\sigma,x,a}$. Such a family will be referred to as a *decomposer*. We define onto it the following transfer quantities between a solution σ and a valuation v :

$$T_{F,\sigma \rightarrow v} = \prod_{x \in X} \delta_{F,\sigma,x,v(x)}. \quad (4)$$

Lemma 7 (Transfer Lemma). Let F be a CSP instance and σ any of its solutions. If w_F is decomposed by family $(\delta_{F,\sigma,x,a})$, then

$$W_F(\sigma) = \sum_{v \in D^X} T_{F,\sigma \rightarrow v}. \quad (5)$$

Proof. It is sufficient to expand the weight of a solution as follows:

$$\begin{aligned}
W_F(\sigma) &= \prod_{x \in X} w_F(\sigma, x) \\
&= \prod_{x \in X} \sum_{a \in D} \delta_{F,\sigma,x,a} \\
&= \sum_{v \in D^X} \prod_{x \in X} \delta_{F,\sigma,x,v(x)} \\
&= \sum_{v \in D^X} T_{F,\sigma \rightarrow v}. \quad \square
\end{aligned}$$

We want to insure that transfers made towards a valuation are at least its unladen weight, hence we define the following property of covering.

Definition 8. Let S be a subset of $\mathcal{S}(F)$; we say that (T_F, S) *covers* U_F iff $\forall v \in D^X$, $\sum_{\sigma \in S} T_{F,\sigma \rightarrow v} \geq U_F(v)$.

Later the set S will be chosen depending on the needs of each application of the weighting technique. But now we can state some general conditions that are sufficient for a weighting scheme to be correct.

5.3. Weight Conservation Theorem

Theorem 9 (Weight Conservation Theorem). Let S be a subset of $\mathcal{S}(F)$. If the following assumptions hold:

1. the weighting seed s_F is unitary,
2. the actual weight w_F is decomposed by the family $(\delta_{F,\sigma,x,a})$ such that (T_F, S) covers U_F ,

then $W_F(S) \geq 1$.

Proof. Since w_F is decomposed by the family $(\delta_{F,\sigma,x,a})$, Lemma 7 asserts that $\forall \sigma \in S$, $W_F(\sigma) = \sum_{v \in D^X} T_{F,\sigma \rightarrow v}$. Thus

$$\begin{aligned}
W_F(S) &= \sum_{\sigma \in S} W_F(\sigma) \\
&= \sum_{\sigma \in S} \sum_{v \in D^X} T_{F,\sigma \rightarrow v} \quad \text{by Lemma 7} \\
&= \sum_{v \in D^X} \sum_{\sigma \in S} T_{F,\sigma \rightarrow v} \\
&\geq \sum_{v \in D^X} U_F(v) \quad \text{since } (T_F, S) \text{ covers } U_F.
\end{aligned}$$

Moreover by Lemma 5, since s_F is unitary, $\sum_{v \in D^X} U_F(v) = 1$. \square

Thus we have exhibited three sufficient conditions to get a weight conservation theorem. These conditions might not be necessary; however not any weighting system w_F will be correct, as shown in the example on Fig. 2(a). So let us introduce a way to build w_F from s_F in a way that is intended to match the conditions of our Weight Conservation Theorem.

5.4. Generators

All weights we put onto solutions (either in Section 5.5 or in Section 6) are built from a *weight generator*, as follows.

Definition 10. A *generator* is a function $\omega_F : X \times D \times \mathcal{P}(D) \rightarrow \mathbf{R}^+$. We say that ω_F is *unitary* iff for every variable x and every nonempty subset Δ of D , $\sum_{a \in \Delta} \omega_F(x, a, \Delta) = 1$.

From the weight generator ω_F we now define the *actual weight* w_F of a variable in a solution:

$$w_F(\sigma, x) = \omega_F(x, \sigma(x), A_F(\sigma, x)). \quad (6)$$

Remark. If σ and τ are 2 solutions such that $\sigma(x) = \tau(x)$ and $A_F(\sigma, x) = A_F(\tau, x)$, then $w_F(\sigma, x) = w_F(\tau, x)$. This is what Boufkhad and Hugel [8] call a *uniform weighting*.

This may suggest that it could be sufficient to put any weights such that the sum of weights on any maximal clique would be 1; but it is not the case (cf. example on Fig. 2(a)).

5.5. Dispatchers

Definition 11. A *dispatcher* is a function $d_F : X \times D \rightarrow \mathbf{R}_+^+$.

Using the weighting seed s_F and the dispatcher d_F we now build the weight generator ω_F of variables in a solution. Each variable will keep its seed s_F ; moreover the weights of forbidden values will be dispatched to allowed values thanks to d_F , in the following way:

$$\omega_F(x, a, \Delta) = \begin{cases} s_F(x, a) + \frac{d_F(x, a)}{\sum_{b \in \Delta} d_F(x, b)} \sum_{b \in D \setminus \Delta} s_F(x, b) & \text{if } a \in \Delta; \\ 0 & \text{otherwise.} \end{cases} \quad (7)$$

Δ represents a category of the set of allowed values; so the dispatcher d_F dispatches the total weighting seed of forbidden values among allowed values.

Fact. If s_F is unitary, so is ω_F .

Definition 12. We say that the weighting system is *homogeneous* when $d_F = s_F$. In this noticeable case the same function is used to assign a weighting seed and to dispatch the remaining weights among neighbors.

Examples of weightings. As one can see in Fig. 2(a), even if we put a total weight of 1 on each clique, the overall weight can be less than 1. To prevent such bad configurations we let our weights take the form of seeds + dispatchers (Fig. 2(b) and (c)). The purpose of building weights from seeds and dispatchers is to prevent the same kind of inversions that we encountered for orientations (which led to circuits): in Fig. 2(a), in clique $\{a, b\}$ for variable x , a is given a much smaller weight than b , whereas in clique $\{a, b, c\}$ for the same variable x , the opposite occurs. In fact dispatchers allow some reshuffling of weights between different cliques of the same variable (see the weights given to values a and b in cliques $\{a, b\}$ and $\{a, b, c\}$ for variable x on Fig. 2(c)), but the fact that seeds and dispatchers are assigned to each individual couple (variable, value) enables a kind of consistency between a clique and its sub-cliques, preventing circuit-like structures from appearing.

- Fig. 2(b) was obtained by the following choice of s_F and d_F (homogeneous case, so $d_F = s_F$):

| s_F | a | b | c |
|-------|-----|-----|-----|
| x | 0.1 | 0.2 | 0.7 |
| y | 0.4 | 0.3 | 0.3 |

| d_F | a | b | c |
|-------|-----|-----|-----|
| x | 0.1 | 0.2 | 0.7 |
| y | 0.4 | 0.3 | 0.3 |

- Fig. 2(c) was obtained by the following choice of s_F and d_F :

| s_F | a | b | c |
|-------|-----|-----|-----|
| x | 0.1 | 0.2 | 0.7 |
| y | 0.4 | 0.3 | 0.3 |

| d_F | a | b | c |
|-------|-----|-----|-----|
| x | 0.6 | 0.3 | 0.1 |
| y | 0.2 | 0.5 | 0.3 |

We come back to our weighting system w_F built from s_F and d_F and show that it may be used to estimate satisfiability if s_F is unitary. So our first result concerning this weighting system states that this system is correct for the estimation of satisfiability (Theorem 15). To prove it, we use our Weight Conservation Theorem, using the following decomposers:

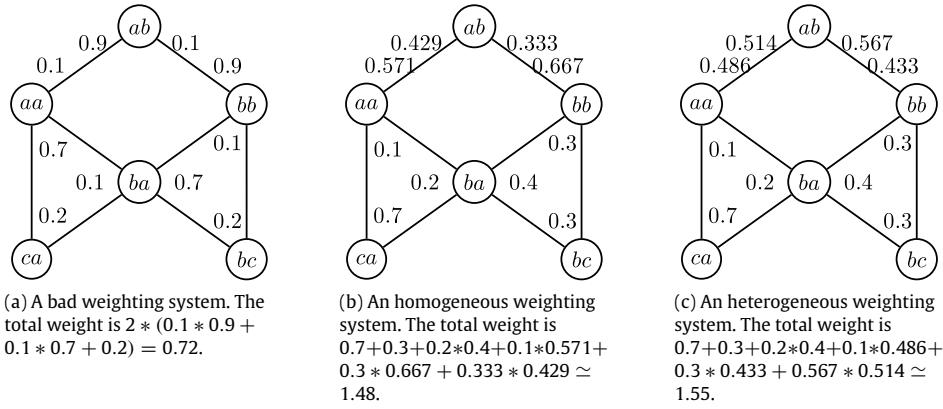


Fig. 2. Some basic examples of weights. Notations are the same as in Fig. 1.

$$\delta_{F, \sigma, x, a} = \begin{cases} s_F(x, a) & \text{if } \sigma(x) = a; \\ \frac{d_F(x, \sigma(x))}{\sum_{a \in A_F(\sigma, x)} d_F(x, a)} s_F(x, a) & \text{if } a \notin A_F(\sigma, x); \\ 0 & \text{otherwise.} \end{cases} \quad (8)$$

We must now prove that the conditions of our Weight Conservation Theorem are satisfied: w_F is decomposed by family $(\delta_{F, \sigma, x, a})$ and (T_F, g) covers U_F .

Lemma 13. w_F is decomposed by family $(\delta_{F, \sigma, x, a})$.

Proof. By definitions:

$$\begin{aligned} \sum_{a \in D} \delta_{F, \sigma, x, a} &= \sum_{a \in D} \left(s_F(x, a) \mathbf{1}_{a=\sigma(x)} + \frac{d_F(x, \sigma(x))}{\sum_{a \in A_F(\sigma, x)} d_F(x, a)} s_F(x, a) \mathbf{1}_{a \notin A_F(\sigma, x)} \right) \\ &= s_F(x, \sigma(x)) + \frac{d_F(x, \sigma(x))}{\sum_{a \in A_F(\sigma, x)} d_F(x, a)} \sum_{a \notin A_F(\sigma, x)} s_F(x, a) \\ &= w_F(\sigma, x). \quad \square \end{aligned}$$

As the unladen weight of a valuation is scattered among lots of solutions, in the proof of the following lemma we use an algorithm building a tree in order to catch enough solutions to insure the covering condition. The proof is somewhat technical and may be skipped at first reading.

Lemma 14. Let g be any connected component of the solutions network $\mathcal{S}(F)$. Then (T_F, g) covers U_F .

Proof. First we need some definitions. A *partial valuation* η over $Y \subseteq X$ is a function from Y to the set D . The *domain* of η is $\text{Dom}(\eta) = Y$. The *level* (of undetermination) of η is $\text{Level}(\eta) = |X \setminus Y|$. Let $Z \subseteq Y \subseteq X$, let ι be a partial valuation over Z and η be a partial valuation over Y . Since $Z \subseteq Y$, we say that $\iota \leq_{\text{Dom}} \eta$. Of course \leq_{Dom} is a partial order relation. We say that η is an *extension* of ι iff $\forall z \in Z, \eta(z) = \iota(z)$, in which case we also say that ι is the restriction of η to Z : $\iota = \eta|_Z$. In the particular case when $Y = Z \cup \{x\}$ with $x \notin Z$, we denote by $\iota_{x \mapsto a}$, the extension of ι to Y assigning value a to x . Let g be a connected component of the solutions network. Note that the empty valuation ϵ (with domain \emptyset) is extensible to a solution in g as soon as $g \neq \emptyset$. Given a partial valuation η , we call $E_g(\eta)$ the set of its extensions which are elements of g and $r_g(\eta)$ the set of restrictions of η extensible to a solution in g (i.e. restrictions r of η such that $E_g(r) \neq \emptyset$).

Let us take any valuation v . We must prove that $\sum_{\sigma \in g} T_{F, \sigma \rightarrow v} \geq U_F(v)$. Since $g \neq \emptyset, \epsilon \in r_g(v)$ so $r_g(v) \neq \emptyset$ and we can pick an element v_0 in $r_g(v)$ maximal with respect to the order \leq_{Dom} . We arbitrarily put indices $1, \dots, n_0$ onto the remaining $n_0 = \text{Level}(v_0)$ variables: x_1, \dots, x_{n_0} (i.e. variables not set by v_0). In the following algorithm we shall bind a *fictitious weight* $f(\eta)$ and a solution $\tau(\eta)$ to a partial valuation η . At the beginning $f(v_0) = U_F(v)$, and we make a call of $\text{Extend}(v_0)$.

Informally we are building a tree and propagating weights from the root v_0 (at level n_0) to leaves which are solutions (at level 0) in a conservative way: the total fictitious weight on level i will be the same as that of level $i + 1$.

Formally, what can we ensure along this process?

1. The first thing to notice is that the algorithm stops; namely the nested calls of $\text{Extend}(\eta)$ decrement $\text{Level}(\eta)$ till it reaches 0.

Algorithm 1 Extensions of a partial valuation.

```

1: procedure Extend( $\eta$ )
2:    $i \leftarrow \text{Level}(\eta)$ 
3:   if  $i = 0$  then
4:      $S \leftarrow S \cup \{\eta\}$ 
5:   else
6:      $\tau(\eta) \leftarrow$  a solution maximizing  $\sum_{b \in A_F(\sigma, x_i)} d_F(x_i, b)$  among  $\sigma \in E_g(\eta)$ 
7:     for all  $a \in A_F(\tau(\eta), x_i)$  do
8:        $f(\eta_{x_i \mapsto a}) \leftarrow \frac{d_F(x_i, a)}{\sum_{b \in A_F(\tau(\eta), x_i)} d_F(x_i, b)} f(\eta)$ 
9:     Extend ( $\eta_{x_i \mapsto a}$ )

```

2. Secondly S is indeed a set of solutions in g extending v_0 . Namely at each call of $\text{Extend}(\eta)$, η is extensible to a solution in g and the set of unset variables of η is $\{x_1, \dots, x_i\}$, where $i = \text{Level}(\eta)$. Thus when $i = 0$, η is a solution in g . We prove this by induction:

- (a) at the beginning: $v_0 \in r_g(v)$, v_0 trivially extends itself, $E_g(v_0) \neq \emptyset$ and the set of unset variables of v_0 is $\{x_1, \dots, x_{n_0}\}$;
 - (b) now suppose that $E_g(\eta) \neq \emptyset$, η extends v_0 and the unset variables of η are $\{x_1, \dots, x_i\}$; given $\tau(\eta) \in E_g(\eta)$, let $a \in A_F(\tau(\eta), x_i)$; then the valuation $\tau(\eta)_{x_i \leftarrow a}$ is a solution by definition of $A_F(\tau(\eta), x_i)$; moreover it is connected to $\tau(\eta)$, thus $\tau(\eta)_{x_i \leftarrow a}$ is an element of component g . Moreover since $\tau(\eta)$ is an extension of η and x_i is unset in η , $\tau(\eta)_{x_i \leftarrow a}$ is an extension of $\eta_{x_i \mapsto a}$. Thus $\tau(\eta)_{x_i \leftarrow a} \in E_g(\eta_{x_i \mapsto a})$, so $E_g(\eta_{x_i \mapsto a}) \neq \emptyset$. Of course, $\eta_{x_i \mapsto a}$ extends v_0 , the unset variables of $\eta_{x_i \mapsto a}$ are $\{x_1, \dots, x_{i-1}\}$ and $\text{Level}(\eta_{x_i \mapsto a}) = \text{Level}(\eta) - 1 = i - 1$.
3. $\sum_{\sigma \in S} f(\sigma) = U_F(v)$; namely among partial valuations considered in the process, $\eta \in S$ iff $\text{Level}(\eta) = 0$. Moreover we now prove by induction that $\sum_{\text{Level}(\eta)=i} f(\eta) = U_F(v)$:
- (a) at the beginning when $i = n_0$, the only partial valuation of level n_0 is v_0 and $f(v_0) = U_F(v)$;
 - (b) now suppose that $\sum_{\text{Level}(\eta)=i} f(\eta) = U_F(v)$; in our process each partial valuation η of level $i - 1$ has one and only one parent in level i , which is given by the restriction η' of η to $\text{Dom}(v_0) \cup \{x_{i+1}, \dots, x_{n_0}\}$; thus

$$\begin{aligned}
 \sum_{\text{Level}(\eta)=i-1} f(\eta) &= \sum_{\text{Level}(\eta')=i} \sum_{a \in A_F(\tau(\eta'), x_i)} f(\eta'_{x_i \mapsto a}) \\
 &= \sum_{\text{Level}(\eta')=i} \sum_{a \in A_F(\tau(\eta'), x_i)} \frac{d_F(x_i, a)}{\sum_{b \in A_F(\tau(\eta'), x_i)} d_F(x_i, b)} f(\eta') \\
 &= \sum_{\text{Level}(\eta')=i} f(\eta') \\
 &= U_F(v).
 \end{aligned}$$

- 4. $\forall \sigma \in S, \forall i \in \{1, \dots, n_0\}, v(x_i) \notin A_F(\sigma, x_i)$. Suppose on the contrary that $\exists \sigma \in S, \exists i \in \{1, \dots, n_0\}, v(x_i) \in A_F(\sigma, x_i)$; the partial valuation $v_{0, x_i \mapsto v(x_i)}$ is still a restriction of v ; moreover, since by item 2, σ is an extension of v_0 , $\sigma_{x_i \leftarrow v(x_i)}$ is an extension of $v_{0, x_i \mapsto v(x_i)}$; and since $v(x_i) \in A_F(\sigma, x_i)$, $\sigma_{x_i \leftarrow v(x_i)}$ is a solution. Thus $v_{0, x_i \mapsto v(x_i)} \in r_g(v)$ and $v_{0, x_i \mapsto v(x_i)} > \text{Dom } v_0$, contradicting the maximality of v_0 in $r_g(v)$.
- 5. $\forall \sigma \in S, f(\sigma) \leq T_{F, \sigma \rightarrow v}$; namely, let us take any $\sigma \in S$:

$$\begin{aligned}
 T_{F, \sigma \rightarrow v} &= \prod_{x \in X} \delta_{F, \sigma, x, v(x)} \quad \text{by Definition 6} \\
 &= \prod_{x \in X} S_F(x, v(x)) \left(\mathbf{1}_{v(x)=\sigma(x)} + \frac{d_F(x, \sigma(x))}{\sum_{a \in A_F(\sigma, x)} d_F(x, a)} \mathbf{1}_{v(x) \notin A_F(\sigma, x)} \right) \quad \text{by Eq. (8)} \\
 &= \prod_{x \in \text{Dom}(v_0)} S_F(x, v(x)) \prod_{i=1}^{n_0} \frac{d_F(x_i, \sigma(x_i)) S_F(x_i, v(x_i))}{\sum_{a \in A_F(\sigma, x_i)} d_F(x_i, a)} \quad \text{by item 2 and 4} \\
 &= U_F(v) \prod_{i=1}^{n_0} \frac{d_F(x_i, \sigma(x_i))}{\sum_{a \in A_F(\sigma, x_i)} d_F(x_i, a)} \quad \text{by Definition 4.}
 \end{aligned}$$

Moreover note that

$$f(\sigma) = f(\sigma_{|\text{Dom}(v_0)}) \prod_{i=1}^{n_0} \frac{f(\sigma_{|\text{Dom}(v_0) \cup \{x_i, \dots, x_{n_0}\}})}{f(\sigma_{|\text{Dom}(v_0) \cup \{x_{i+1}, \dots, x_{n_0}\}})}$$

$$\begin{aligned}
&= f(v_0) \prod_{i=1}^{n_0} \frac{f\left(\sigma_{|\text{Dom}(v_0) \cup \{x_{i+1}, \dots, x_{n_0}\}}, x_i \mapsto \sigma(x_i)\right)}{f\left(\sigma_{|\text{Dom}(v_0) \cup \{x_{i+1}, \dots, x_{n_0}\}}\right)} \\
&= U_F(v) \prod_{i=1}^{n_0} \frac{d_F(x_i, \sigma(x_i))}{\sum_{a \in A_F\left(\tau\left(\sigma_{|\text{Dom}(v_0) \cup \{x_{i+1}, \dots, x_{n_0}\}}\right), x_i\right)} d_F(x_i, a)}.
\end{aligned}$$

Since of course, for any i between 1 and n_0 , $\sigma \in E_g\left(\sigma_{|\text{Dom}(v_0) \cup \{x_{i+1}, \dots, x_{n_0}\}}\right)$, by the choice of $\tau(\eta)$ in line 6 of algorithm 1, we have that $\sum_{a \in A_F\left(\tau\left(\sigma_{|\text{Dom}(v_0) \cup \{x_{i+1}, \dots, x_{n_0}\}}\right), x_i\right)} d_F(x_i, a) \geq \sum_{a \in A_F(\sigma, x_i)} d_F(x_i, a)$, whence $f(\sigma) \leq U_F(v)$

$$\prod_{i=1}^{n_0} \frac{d_F(x_i, \sigma(x_i))}{\sum_{a \in A_F(\sigma, x_i)} d_F(x_i, a)} = T_{F, \sigma \rightarrow v}.$$

Thus we finally get that

$$\begin{aligned}
\sum_{\sigma \in g} T_{F, \sigma \rightarrow v} &\geq \sum_{\sigma \in g} f(\sigma) \quad \text{by item 5} \\
&\geq \sum_{\sigma \in S} f(\sigma) \quad \text{because } S \subseteq g.
\end{aligned}$$

Moreover, by item 3, $\sum_{\sigma \in S} f(\sigma) = U_F(v)$; thus (T_F, g) covers U_F . \square

From Lemmas 13 and 14, we conclude that our weighting system built from seeds and dispatchers obeys the conditions of the Weight Conservation Theorem.

Theorem 15. Let F be a satisfiable CSP instance and g the solutions in a connected component of the solutions network of F . Weights w_F are built from seeds s_F and dispatchers d_F , as in Definition 11. If the weighting seed s_F is unitary, then $W_F(g) \geq 1$.

Remark. In this paper we do not address the question of choosing the best s_F and d_F for a given instance F or for a given family of instances, which must be custom-tailored depending on the considered problem.

Remark. As a consequence, each connected component has at least weight 1. Thus the total weight of the solution network is at least the number of its connected components. Being based on the notion of neighborhood, both methods of local weighting and ordering are subject to this downside.

Seeds and dispatchers yield a general non-trivial weighting scheme which is suitable for the first moment method; we are now ready to introduce a slight variation on them and apply it to the measurement of Boolean cores.

6. Boolean case: a better upper bound on the existence of non-trivial cores

6.1. A better weighting for trivaluations

In order to estimate Boolean satisfiability of formulas, Maneva and Sinclair [17] use a so called *Weight Preservation Theorem*. Valuations here are mappings from X to $D = \{0, 1, *\}$, so we call them trivaluations or for short valuations. The value $*$ is to be interpreted as 0 or 1. They call such a valuation *valid* iff each clause contains at least one true literal or two starred literals. In this section, one has to be aware about the fact that we define a *Boolean solution* as a valid valuation taking its values in $\{0, 1\}$ only! $\mathcal{S}(F)$ still denotes the set of valid valuations of an instance F (with values in $\{0, 1, *\}$) and $A_F(\sigma, x)$ still refers to a neighborhood in $\mathcal{S}(F)$. Note that any formula has at least one valid valuation: the one that gives the value $*$ to every variable (the so-called trivial core), so the existence of valid valuations does not guarantee the existence of Boolean solutions. Nevertheless, counting weighted valid valuations can be used to estimate Boolean satisfiability.

Maneva and Sinclair choose their weights as follows: each variable has a weighting seed $s_0(x)$, $s_*(x)$ such that $s_0(x) + s_*(x) = 1$, and for every valid valuation σ and every variable x they put the following weight:

$$q_F(\sigma, x) = \begin{cases} s_*(x) & \text{if } \sigma(x) = *; \\ s_0(x) & \text{if } \sigma(x) \neq * \text{ and } * \in A_F(\sigma, x); \\ s_0(x) + s_*(x) & \text{otherwise.} \end{cases} \quad (9)$$

As shown by Maneva et al. [16], the sum of the weights of all valid valuations reachable from any Boolean solution is exactly 1. The reachability property is defined as the existence of a path from the Boolean solution to the valid valuation where at each step a variable is given the value $*$ while maintaining the validity property. Since a given valid valuation may be reachable from lots of different Boolean solutions (but sometimes from no one), counting the weighted valid trivaluations hopefully enables us to count less than the number of Boolean solutions.

Using these weights, Maneva and Sinclair [17] count the so called non-trivial cores σ ; a non-trivial core σ is a valid valuation with a linear number of non-starred and non-starrable variables (i.e. such that $A_F(\sigma, x) = \{\sigma(x)\}$). Many non-trivial cores are not extensible to solutions; a core is extensible to a solution when there is a Boolean valuation of the starred literals which is a Boolean solution. They manage to count only cores which are extensible to a Boolean solution, and they estimate the satisfiability of the starred part of the formula by weighting valid assignments as defined in Eq. (9). In this section we define our new weights and show that they are correct, and in Section 6.2 we use them to improve on Maneva and Sinclair's upper bound, from 4.453 to 4.419.

Before we give this improvement and show its correctness, we want to stress an important difference between the weighting of solutions of general CSPs as defined in the previous sections and the weighting defined in this section: in the previous sections, an unsatisfiable formula has always a total weight of 0 while in the present one, an unsatisfiable formula (a formula with no Boolean solution) will have a non-zero total weight (provided the weights of the value $*$ are not 0). This is the price one has to pay to lower the weights of satisfiable formulas. This fact makes it difficult to establish a general comparison between both methods, because they are highly dependent on the set of instances that are considered and in particular on the proportion of unsatisfiable instances among them.

Remarkably, a sufficient condition ensuring the covering condition of the Weight Conservation Theorem will be that the subset S of valid valuations contains a Boolean solution.

To improve on Maneva et al.'s estimation system, we choose the following weights: each variable x has a unitary weighting seed $s_F(x, 0)$, $s_F(x, 1)$ and $s_F(x, *)$. From this seed s_F we define the weight generator ω_F as follows:

$$\omega_F(x, a, \Delta) = \begin{cases} s_F(x, a) & \text{if } a = * \text{ and } a \in \Delta; \\ s_F(x, a) + \sum_{b \in D \setminus \Delta} s_F(x, b) & \text{if } a \neq * \text{ and } a \in \Delta; \\ 0 & \text{if } a \notin \Delta. \end{cases} \quad (10)$$

As before in Section 5.5, we define the actual weight $w_F(\sigma, x) = \omega_F(x, \sigma(x), A_F(\sigma, x))$.

Remark 16. Noticeable values of ω_F :

$$\begin{aligned} \omega_F(x, 0, \{0\}) &= \omega_F(x, 1, \{1\}) = s_F(x, 0) + s_F(x, 1) + s_F(x, *) = 1; \\ \omega_F(x, 0, \{0, *\}) &= \omega_F(x, 1, \{1, *\}) = s_F(x, 0) + s_F(x, 1); \\ \omega_F(x, a, \{0, 1, *\}) &= s_F(x, a); \\ \omega_F(x, *, \Delta) &= s_F(x, *) \quad \text{if } * \in \Delta. \end{aligned}$$

Remark 17. ω_F is almost unitary, since for any nonempty $\Delta \subseteq D$, $\Delta \neq \{*\}$, $\Delta \neq \{0, 1\}$, $\sum_{a \in \Delta} \omega(x, a, \Delta) = 1$; $\{0, 1\}$ cannot be a clique in this model of validity, because if both 0 and 1 are allowed, so is $*$. However $\{*\}$ can be a clique, and in this case $\omega(x, *, \{*\}) = s_F(x, *)$.

Our system can be seen as a split of $1 - s_*(x)$ into $s_F(x, 0)$ and $s_F(x, 1)$ (instead of just $s_0(x)$ for Maneva) in the case when $\sigma(x) \neq *$ and $A_F(\sigma, x) = \{0, 1, *\}$; thus our weights are smaller than Maneva's, though we are able to insure that they are correct.

This system is different from the system seeds+dispatchers, because here a fixed variable at value $*$ is given a weight of $s_F(x, *)$, whereas dispatchers would give it a weight of 1. However we are able to use our Weight Conservation Theorem, using the following decomposers:

$$\delta_{F, \sigma, x, a} = \begin{cases} s_F(x, a) & \text{if } \begin{cases} \sigma(x) = a \\ \text{or } (\sigma(x) \neq * \text{ and } a \notin A_F(\sigma, x)); \end{cases} \\ 0 & \text{otherwise.} \end{cases} \quad (11)$$

We must now prove that the conditions of our Weight Conservation Theorem are satisfied: w_F is decomposed family $(\delta_{F, \sigma, x, a})$ and (T_F, g) covers U_F .

Lemma 18. w_F is decomposed by family $(\delta_{F, \sigma, x, a})$.

Proof. By definitions:

1. if $\sigma(x) = *$: $\sum_{a \in D} \delta_{F, \sigma, x, a} = \sum_{a \in D} s_F(x, a) \mathbf{1}_{a=\sigma(x)} = s_F(x, \sigma(x)) = w_F(\sigma, x)$;
2. if $\sigma(x) \neq *$:

$$\begin{aligned} \sum_{a \in D} \delta_{F, \sigma, x, a} &= \sum_{a \in D} s_F(x, a) (\mathbf{1}_{a=\sigma(x)} + \mathbf{1}_{a \notin A_F(\sigma, x)}) \\ &= s_F(x, \sigma(x)) + \sum_{b \notin A_F(\sigma, x)} s_F(x, b) \\ &= w_F(\sigma, x). \quad \square \end{aligned}$$

Lemma 19. Let v be a valuation and g be any connected component of the network of valid valuations $\mathcal{S}(F)$ containing a Boolean solution. Then there exists a valid valuation $\sigma \in g$ such that $U_F(v) = T_{F,\sigma \rightarrow v}$.

Proof. Let us take any $v \in D^X$ and a Boolean solution $\sigma_0 \in g$. At the beginning we put $\sigma = \sigma_0$. Consider the following procedure:

- If there is a variable $x \in X$ such that $\sigma(x) \neq v(x)$ and $\sigma_{x \leftarrow v(x)}$ remains a valid valuation, then change σ to $\sigma_{x \leftarrow v(x)}$.

We iterate this procedure till there is no variable $x \in X$ such that $\sigma(x) \neq v(x)$ and $\sigma_{x \leftarrow v(x)}$ remains a valid valuation. This eventually happens because at each step we make a move towards v , and X is finite. So in the end, each variable in σ has either its initial Boolean value in σ_0 or the value given by v . In other words, for any $x \in X$, either $\sigma(x) = v(x)$ or $(\sigma(x) \neq * \text{ and } v(x) \notin A_F(\sigma, x))$. Thus by Eq. (11), $\delta_{F,\sigma,x,v(x)} = s_F(x, v(x))$, which in turn by Definitions 4 and 6 yields $T_{F,\sigma \rightarrow v} = U_F(v)$. Moreover, by construction, the ending σ is also in g . \square

Corollary 20. Let g be any connected component of the network of valid valuations $\mathcal{S}(F)$ containing a Boolean solution. Then (T_F, g) covers U_F .

Thus our weighting system obeys the Weight Conservation Theorem, and we can conclude that $\gamma(F) = W_F(\mathcal{S}(F)) \geq W_F(g) \geq 1$ and state the following theorem:

Theorem 21. w_F as defined in Eq. (10) yields $\gamma(F) \geq 1$ whenever F admits a Boolean solution.

Note that in the particular case where for any $x \in X$, $s_F(x, 0) = 0$, $s_F(x, 1) = 1$ and $s_F(x, *) = 0$, we count what Dubois and Boufkhad [10] call Negatively Prime Solutions (NPSs). Moreover, as soon as $s_F(x, *) = 0$, this weighting can be seen as seeds + dispatchers on a Boolean domain (so this weighting is homogeneous).

We used the weighting defined in Eq. (10) to compute an upper bound of the threshold of random 3-SAT: taking seeds independent of F and x , we obtained the best estimation when $s_F(x, *) = 0$; the corresponding upper bound is 4.643, which is precisely the NPS-based upper-bound of Dubois and Boufkhad [10]. We conjecture that even if one takes seeds dependent on F or x , the best choice of $s_F(x, *)$ to estimate Boolean unsatisfiability remains indeed 0. The reason why we think so, is that, as described in Remark 17, ω_F is almost unitary, except for the clique $\{*\}$, in which case $\omega_F(x, *, \{*\}) = s_F(x, *)$.

6.2. Application: non-existence of non-trivial cores in 3-SAT

We apply here the weighting defined in 6.1 to improve on the upper bound on the existence of non-trivial cores in 3-SAT shown by Maneva and Sinclair [17]. First, we recall the basic notions defined by Maneva and Sinclair [17] and we reformulate them according to our notation. Starting from solutions (which are some kind of valid trivaluations), the following process is iterated: whenever there is a starrable variable x (i.e. a variable such that under the current valuation σ , we have $\sigma(x) \neq *$ and $A(\sigma, x)$ contains $*$), consider for the next step the valuation obtained from σ by assigning $*$ to x . This process stops when no such variables exist, and the resulting trivaluation is called a *core*. Only the cores σ for which the set of non-starred variables is not empty are interesting; they are said to be non-trivial. Such cores contain important information on the geometry of the space of solutions and underlie the so-called clustering that explains the difficulty of solving such instances.

To study the existence of cores, Maneva and Sinclair [17] make use of *covers*. A cover is a trivaluation where every non-starred variable are non-starrable. Obviously, each core is also a cover but the converse is not true. Indeed, by construction, for a core there exist always a way of assigning the starred variables so that the formula is satisfied, while for covers no such condition is guaranteed. The size of a core or a cover is the number of non-starred variables.

Maneva and Sinclair [17] compute their upper bound in two steps:

1. First, they use the first moment method to upper-bound the probability of existence of covers of certain sizes. This allows them to discard the ranges of sizes where covers do not exist, and within these ranges cores cannot exist either;
2. As for the ranges that are not discarded by the previous step, the probability that covers can be extended to solutions is upper-bounded using the first moment method through the weighting system.

Implementing this method as described in [17] we obtain the following improvement.

Theorem 22. Random instances of 3-SAT with density greater than 4.419 have no non-trivial cores with high probability.

As we mimic Maneva and Sinclair [17]'s proof, we defer it to Appendix. Let us just mention the weights we use: $s_F(x, *) = \rho$, $s_F(x, 1) = s_F(x, 0) = (1 - \rho)/2$. We determined the best values for ρ by numerical simulations, and it depends on the parameter a (size of cores). By symmetry, any combination of values given to the $s_F(x, 1)$ and $s_F(x, 0)$ summing up to $1 - \rho$ will give the same result.

One can wonder whether this bound can be improved upon using Boolean solutions and ordering. The answer is no since that case is a particular case of our weighting system (Eq. (10)): the case where $s_F(x, *) = 0$, $s_F(x, 1) = 1$ and $s_F(x, 0) = 0$ or vice-versa. It is easy to answer this question in the Boolean case because any ordering can be seen as a particular weighting (by choosing the weight 1 for a value and 0 for the other one). But in the general case of larger domain CSPs the answer is not so easy. In the following section, we give general comparisons between orderings and weightings in different general cases.

7. Weighting versus ordering

7.1. Homogeneous case: homogeneous weighting is not better than ordering

As we have seen, the weighting is based upon two functions:

1. the weighting seed s_F that determines the intrinsic weight of each value and then allows us to compute the intrinsic unladen weight of each valuation;
2. the dispatcher d_F that represents how the weights of forbidden valuations are scattered among the authorized ones.

A natural case to investigate is when these two quantities are equal, namely when each allowed value is dispatched a complimentary weight proportional to its intrinsic weight. So we deal here with the homogeneous case $d_F = s_F$ and show that whatever s_F may be, there will exist an ordering which is at least as good as the weighting system, as will be stated in [Theorem 32](#). The proof consists in choosing variable per variable the order $<_{F,x}$ in a way that does not increase the global weight. For our recurrence to work we use the homogeneity property. Just as we defined a generator ω_F for a weight w_F , so we now need to define a generator μ_F for an orientation m_F .

Definition 23. We define the following binary weight function:

$$\mu_F(x, a, \Delta) = \begin{cases} 1 & \text{if } a \text{ is the minimum of } \Delta \text{ for } <_{F,x}; \\ 0 & \text{otherwise.} \end{cases} \quad (12)$$

$$m_F(\sigma, x) = \mu_F(x, \sigma(x), A_F(\sigma, x)). \quad (13)$$

At each step of the recurrence, some variables are ordered while the other ones are weighted. That leads us to introduce the following definitions. We are going to substitute binary weights m_F 's to original weights w_F 's variable per variable, so we call \mathcal{E} the set of couples of (variables x , orders $<_{F,x}$) where m_F 's are used and we define

Definition 24.

$$\Omega_F(\sigma, \mathcal{E}) = \prod_{x \in \mathcal{E}} m_F(\sigma, x) \prod_{x \in X \setminus \mathcal{E}} w_F(\sigma, x) \quad (14)$$

and we extend it to a set S of solutions by

$$\Omega_F(S, \mathcal{E}) = \sum_{\sigma \in S} \Omega_F(\sigma, \mathcal{E}). \quad (15)$$

Remark 25. What happens when \mathcal{E} is empty?

$$\Omega_F(\mathcal{S}(F), \emptyset) = W_F(\mathcal{S}(F)). \quad (16)$$

Namely, by definition, for any solution $\sigma \in \mathcal{S}(F)$, $\Omega_F(\sigma, \emptyset) = \prod_{x \in X} w_F(\sigma, x) = W_F(\sigma)$. But $\Omega_F(\mathcal{S}(F), \emptyset) = \sum_{\sigma \in \mathcal{S}(F)} \Omega_F(\sigma, \emptyset)$ and $W_F(\mathcal{S}(F)) = \sum_{\sigma \in \mathcal{S}(F)} W_F(\sigma)$.

Remark 26. What happens when \mathcal{E} is full? Suppose that for every variable x , $<_{F,x}$ is a total order on D . Let $<_F$ be the orientation induced by $\{(x, <_{F,x})\}_{x \in X}$. Then

$$\Omega_F(\mathcal{S}(F), \{(x, <_{F,x})\}_{x \in X}) = |\mathcal{M}_{<_F}(F)|. \quad (17)$$

Namely, let us recall that for any solution $\sigma \in \mathcal{S}(F)$, $\Omega_F(\sigma, \{(x, <_{F,x})\}_{x \in X}) = \prod_{x \in X} m_F(\sigma, x)$. Thus $\Omega_F(\sigma, \{(x, <_{F,x})\}_{x \in X}) = 1$ iff $\forall x \in X$, σ is the minimum of $N_F(\sigma, x)$ for $<_{F,x}$ (or equivalently for $<_F$); in other words σ is minimal among all of its neighbors, which means that σ is minimal (since $<_F$ compares neighboring solutions only). Thus $\Omega_F(\{(x, <_{F,x})\}_{x \in X})$ is the number of minimal elements of the underlying orientation $<_F$.

We are now ready to state the main lemma in this section.

Lemma 27. Suppose that s_F is unitary and $d_F = s_F$. Then for each set \mathcal{E} , each variable $x_0 \notin \mathcal{E}$, there exists a total order $<_{F,x_0}$ on D such that $\Omega_F(\mathcal{S}(F), \mathcal{E} \cup \{(x_0, <_{F,x_0})\}) \leq \Omega_F(\mathcal{S}(F), \mathcal{E})$.

At first reading it might be convenient to jump directly to [Theorem 32](#), because the proof of [Lemma 27](#) is somewhat technical and requires some more notation and sub-lemmas. We fix a variable $x_0 \notin \mathcal{E}$. Let a be an element of D and Δ be a subset of D . We consider the preimages of (a, Δ) obtained through mapping a solution σ of instance F to $(\sigma(x_0), A_F(\sigma, x_0))$. We denote these preimages as follows:

$$\Sigma_{F,x_0}(a, \Delta) = \{\sigma \in \mathcal{S}(F), \sigma(x_0) = a \text{ and } A_F(\sigma, x_0) = \Delta\}. \quad (18)$$

Note that:

1. when $a \notin \Delta$, $\Sigma_{F,x_0}(a, \Delta) = \emptyset$;
2. the $\Sigma_{F,x_0}(a, \Delta)$ are pairwise disjoint and $\bigsqcup_{\substack{\Delta \subseteq D \\ a \in \Delta}} \Sigma_{F,x_0}(a, \Delta) = \mathcal{F}(F)$;
3. if $\sigma, \tau \in \Sigma_{F,x_0}(a, \Delta)$, then $w_F(\sigma, x_0) = w_F(\tau, x_0) = \omega_F(x_0, a, \Delta)$ and $m_F(\sigma, x_0) = m_F(\tau, x_0) = \mu_F(x_0, a, \Delta)$;
4. we call

$$Z_{F,\mathcal{E},x_0}(a, \Delta) = \sum_{\sigma \in \Sigma_{F,x_0}(a, \Delta)} \prod_{x \in \mathcal{E}} m_F(\sigma, x) \prod_{x \in X \setminus (\mathcal{E} \cup \{x_0\})} w_F(\sigma, x); \quad (19)$$

then by item 3,

$$\Omega_F(\Sigma_{F,x_0}(a, \Delta), \mathcal{E}) = \omega_F(x_0, a, \Delta) \cdot Z_{F,\mathcal{E},x_0}(a, \Delta); \quad (20)$$

$$\Omega_F(\Sigma_{F,x_0}(a, \Delta), \mathcal{E} \cup \{(x_0, <_{F,x_0})\}) = \mu_F(x_0, a, \Delta) \cdot Z_{F,\mathcal{E},x_0}(a, \Delta). \quad (21)$$

We now need to explore further both quantities we want to compare. It will be convenient to use the following quantities: let $E \subseteq D$ and $a \in E$; we define the following quantities:

$$\zeta_{F,\mathcal{E},x_0}(a, E) = \sum_{\substack{\Delta \subseteq E \\ \Delta \ni a}} Z_{F,\mathcal{E},x_0}(a, \Delta); \quad (22)$$

$$\xi_{F,\mathcal{E},x_0}(E) = \sum_{\Delta \subseteq E} \sum_{a \in \Delta} \omega_F(x_0, a, \Delta) \cdot Z_{F,\mathcal{E},x_0}(a, \Delta). \quad (23)$$

So what is the purpose of introducing these extra quantities? They will help us prove [Lemma 27](#) through the following facts.

Fact 28. If $a_1 <_{F,x_0} a_2 <_{F,x_0} \dots <_{F,x_0} a_d$, then

$$\Omega_F(\mathcal{F}(F), \mathcal{E} \cup \{(x_0, <_{F,x_0})\}) = \sum_{i=1}^d \zeta_{F,\mathcal{E},x_0}(a_i, D \setminus \{a_1, \dots, a_{i-1}\}).$$

Proof. We use the partition mentioned in item 2:

$$\begin{aligned} \Omega_F(\mathcal{F}(F), \mathcal{E} \cup \{(x_0, <_{F,x_0})\}) &= \Omega_F\left(\bigsqcup_{\substack{\Delta \subseteq D \\ a \in \Delta}} \Sigma_{F,x_0}(a, \Delta), \mathcal{E} \cup \{(x_0, <_{F,x_0})\}\right) \\ &= \sum_{\substack{\Delta \subseteq D \\ a \in \Delta}} \Omega_F(\Sigma_{F,x_0}(a, \Delta), \mathcal{E} \cup \{(x_0, <_{F,x_0})\}) \\ &= \sum_{\substack{\Delta \subseteq D \\ a \in \Delta}} \mu_F(x_0, a, \Delta) \cdot Z_{F,\mathcal{E},x_0}(a, \Delta) \quad \text{by Eq. (21)} \\ &= \sum_{a \in D} \sum_{\substack{\Delta \subseteq D \\ \Delta \ni a}} \mu_F(x_0, a, \Delta) \cdot Z_{F,\mathcal{E},x_0}(a, \Delta) \\ &= \sum_{a \in D} \sum_{\substack{\Delta \subseteq D \\ \Delta \ni a}} \mathbf{1}_{a \text{ is the minimum of } \Delta \text{ for } <_{F,x_0}} \cdot Z_{F,\mathcal{E},x_0}(a, \Delta) \\ &= \sum_{i=1}^d \sum_{\substack{\Delta \subseteq D \setminus \{a_1, \dots, a_{i-1}\} \\ \Delta \ni a_i}} Z_{F,\mathcal{E},x_0}(a_i, \Delta) \quad \text{since } a_1 <_{F,x_0} \dots <_{F,x_0} a_d \\ &= \sum_{i=1}^d \zeta_{F,\mathcal{E},x_0}(a_i, D \setminus \{a_1, \dots, a_{i-1}\}). \quad \square \end{aligned}$$

Fact 29. For any $x_0 \notin \mathcal{E}$, $\Omega_F(\mathcal{F}(F), \mathcal{E}) = \xi_{F,\mathcal{E},x_0}(D)$.

Proof. We use again the partition mentioned in item 2:

$$\Omega_F(\mathcal{F}(F), \mathcal{E}) = \Omega_F\left(\bigsqcup_{\substack{\Delta \subseteq D \\ a \in \Delta}} \Sigma_{F,x_0}(a, \Delta), \mathcal{E}\right)$$

$$\begin{aligned}
&= \sum_{\substack{\Delta \subseteq D \\ a \in \Delta}} \Omega_F(\Sigma_{F, x_0}(a, \Delta), \mathcal{E}) \\
&= \sum_{\substack{\Delta \subseteq D \\ a \in \Delta}} \omega_F(x_0, a, \Delta) \cdot Z_{F, \mathcal{E}, x_0}(a, \Delta) \quad \text{by Eq. (20)} \\
&= \xi_{F, \mathcal{E}, x_0}(D). \quad \square
\end{aligned}$$

Fact 30. If $E \subseteq D$, $\Delta \subseteq E$, $a \in \Delta$, s_F is unitary and $d_F = s_F$ then

$$\omega_F(x, a, \Delta) \sum_{b \in E} d_F(x, b) = d_F(x, a) + \omega_F(x, a, \Delta) \sum_{b \in E \setminus \Delta} d_F(x, b).$$

Proof. If $a \in \Delta$, then by Eq. (7), $\omega_F(x, a, \Delta) \sum_{b \in \Delta} d_F(x, b) = s_F(x, a) \sum_{b \in \Delta} d_F(x, b) + d_F(x, a) \sum_{b \in D \setminus \Delta} s_F(x, b)$. By equality $d_F = s_F$ and the fact that s_F is unitary, we get $d_F(x, a) = \omega_F(x, a, \Delta) \sum_{b \in \Delta} d_F(x, b)$. \square

Fact 31. Let $x_0 \notin \mathcal{E}$ and E any nonempty subset of D . Suppose that s_F is unitary and $d_F = s_F$. Then there exists $a \in E$ such that $\xi_{F, \mathcal{E}, x_0}(E) \geq \zeta_{F, \mathcal{E}, x_0}(a, E) + \xi_{F, \mathcal{E}, x_0}(E \setminus \{a\})$.

Proof. Let us call a_0 an element of E minimizing $\zeta_{F, \mathcal{E}, x_0}(a, E) + \xi_{F, \mathcal{E}, x_0}(E \setminus \{a\})$ when $a \in E$:

$$\begin{aligned}
\xi_{F, \mathcal{E}, x_0}(E) \sum_{b \in E} d_F(x_0, b) &= \sum_{b \in E} d_F(x_0, b) \sum_{\substack{\Delta \subseteq E \\ a \in \Delta}} \omega_F(x_0, a, \Delta) Z_{F, \mathcal{E}, x_0}(a, \Delta) \\
&= \sum_{\substack{\Delta \subseteq E \\ a \in \Delta}} d_F(x_0, a) Z_{F, \mathcal{E}, x_0}(a, \Delta) + \sum_{\substack{\Delta \subseteq E \\ a \in \Delta}} \omega_F(x_0, a, \Delta) \sum_{b \in E \setminus \Delta} d_F(x_0, b) Z_{F, \mathcal{E}, x_0}(a, \Delta) \quad \text{by Fact 30} \\
&= \sum_{\substack{\Delta \subseteq E \\ a \in \Delta}} d_F(x_0, a) Z_{F, \mathcal{E}, x_0}(a, \Delta) + \sum_{\substack{\Delta \subseteq E \\ a \in E \setminus \Delta \\ b \in \Delta}} \omega_F(x_0, b, \Delta) d_F(x_0, a) Z_{F, \mathcal{E}, x_0}(b, \Delta) \\
&= \sum_{a \in E} d_F(x_0, a) \sum_{\substack{\Delta \subseteq E \\ \Delta \ni a}} Z_{F, \mathcal{E}, x_0}(a, \Delta) + \sum_{a \in E} d_F(x_0, a) \sum_{\Delta \subseteq E \setminus \{a\}} \sum_{b \in \Delta} \omega_F(x_0, b, \Delta) Z_{F, \mathcal{E}, x_0}(b, \Delta) \\
&= \sum_{a \in E} d_F(x_0, a) (\zeta_{F, \mathcal{E}, x_0}(a, E) + \xi_{F, \mathcal{E}, x_0}(E \setminus \{a\})) \\
&\geq \sum_{a \in E} d_F(x_0, a) (\zeta_{F, \mathcal{E}, x_0}(a_0, E) + \xi_{F, \mathcal{E}, x_0}(E \setminus \{a_0\})).
\end{aligned}$$

That gives what we want since $\sum_{b \in E} d_F(x_0, b) \neq 0$ (by Definition 11, dispatchers must be positive). \square

Proof of Lemma 27. By Fact 29, $\Omega_F(\mathcal{J}(F), \mathcal{E}) = \xi_{F, \mathcal{E}, x_0}(D)$. From D we successively remove what we call a_1, a_2, \dots, a_d till we reach the empty set; applying at each step Fact 31 yields that $\xi_{F, \mathcal{E}, x_0}(D) \geq \sum_{i=1}^d \zeta_{F, \mathcal{E}, x_0}(a_i, D \setminus \{a_1, \dots, a_{i-1}\}) + \xi_{F, \mathcal{E}, x_0}(\emptyset)$. By definition, $\xi_{F, \mathcal{E}, x_0}(\emptyset) = 0$. What order $<_{F, x_0}$ shall we choose on D ?

Of course: $a_1 <_{F, x_0} a_2 <_{F, x_0} \dots <_{F, x_0} a_d$.

Then by Fact 28, $\Omega_F(\mathcal{J}(F), \mathcal{E} \cup \{(x_0, <_{F, x_0})\}) = \sum_{i=1}^d \zeta_{F, \mathcal{E}, x_0}(a_i, D \setminus \{a_1, \dots, a_{i-1}\})$. So in the end $\Omega_F(\mathcal{J}(F), \mathcal{E}) \geq \Omega_F(\mathcal{J}(F), \mathcal{E} \cup \{(x_0, <_{F, x_0})\})$. \square

Theorem 32. For any instance F , any positive and unitary weighting seed s_F , when $d_F = s_F$ (homogeneous case), there exists an instance dependent orientation $<_F$ induced by a set $\{(x, <_{F, x})\}_{x \in X}$ of total orders on D , such that $|\mathcal{M}_{<_F}(F)| \leq W_F(\mathcal{J}(F))$.

Proof. By Remark 25, $W_F(\mathcal{J}(F)) = \Omega_F(\mathcal{J}(F), \emptyset)$. Starting with $\mathcal{E} = \emptyset$, we add elements $(x_0, <_{F, x_0})$ to \mathcal{E} such that $\Omega_F(\mathcal{J}(F), \mathcal{E}) \geq \Omega_F(\mathcal{J}(F), \mathcal{E} \cup \{(x_0, <_{F, x_0})\})$, which is possible by Lemma 27. At the end of the process we have thus $\Omega_F(\mathcal{J}(F), \emptyset) \geq \Omega_F(\mathcal{J}(F), \{(x, <_{F, x})\}_{x \in X})$. Let $<_F$ be the orientation on $\mathcal{J}(F)$ induced by $\{(x, <_{F, x})\}_{x \in X}$.

By Remark 26, $\Omega_F(\mathcal{J}(F), \{(x, <_{F, x})\}_{x \in X}) = |\mathcal{M}_{<_F}(F)|$. So $W_F(\mathcal{J}(F)) \geq |\mathcal{M}_{<_F}(F)|$. \square

Whether this theorem is true for heterogeneous weights remains an open question.

Remark. In the particular case of Boolean satisfiability (i.e. when $D = \{0, 1\}$), there is no choice on d_F : the weighting system is necessarily homogeneous. Thus in this case weighting is not better than ordering.

7.2. Instance independent case: ordering and weighting are equivalent

Definition 33. The weight of a CSP instance F is:

$$\gamma(F) = W_F(\mathcal{S}(F)). \quad (24)$$

By extension, the weight of a set \mathcal{F} of CSP instances is:

$$\gamma(\mathcal{F}) = \sum_{F \in \mathcal{F}} \gamma(F). \quad (25)$$

A *permutation* over the domain of values is a bijection $\pi : D \rightarrow D$. A *renaming* of values is a family of permutations $\Pi = (\pi_x)_{x \in X}$ over the domain D . For a CSP instance F , let $\Pi(F)$ be the instance where every occurrence of a value a for every variable x are replaced by $\pi_x(a)$. A set of CSP instances \mathcal{F} is said to be *closed under renaming* if for any renaming Π , if $F \in \mathcal{F}$ then $\Pi(F) \in \mathcal{F}$. By abuse of notation, for any valuation v , we denote by $\Pi(v)$ the valuation that assigns value $\pi_x(v(x))$ to variable x .

Let us first give a very simple yet useful fact:

Fact 34. Let Π be a renaming, F and G be CSP instances. Then

1. $\sigma \in \mathcal{S}(F)$ iff $\Pi(\sigma) \in \mathcal{S}(\Pi(F))$;
2. $A_{\Pi(F)}(\Pi(\sigma), x) = \pi_x(A_F(\sigma, x))$.

Remark. Note that almost all sets of CSP instances we know to be dealt with in the literature are closed under renaming. A counterexample could be a set of Horn formulas. Another one is the trivaluations of Section 6.1: namely $\{0, 1\}$ cannot be a clique (if both 0 and 1 are allowed, so is $*$), whereas $\{0, *\}$ and $\{1, *\}$ can.

Let \mathcal{F} be some set of instances closed under renaming. We prove in the sequel that $\gamma(\mathcal{F}) = \sum_{F \in \mathcal{F}} |\mathcal{M}_{\prec}(F)|$ for any instance independent orientation \prec on solutions as defined in Section 4. That can be interpreted as follows: on average on \mathcal{F} , the weight of all solutions is equal to the number of minimal solutions, independently of the orientation \prec . The proof idea is to partition the couples (solutions, instances) in a way that the weight of each class of the partition has a weight of 1 and corresponds to a minimal element for \prec .

We define the set \mathcal{C} of couples (σ, F) where F is an element of \mathcal{F} and σ a solution of F :

$$\mathcal{C} = \{(\sigma, F) \mid \substack{F \in \mathcal{F} \\ \sigma \in \mathcal{S}(F)}\}. \quad (26)$$

$\gamma(\mathcal{F})$ can be written as

$$\gamma(\mathcal{F}) = \sum_{(\sigma, F) \in \mathcal{C}} W_F(\sigma). \quad (27)$$

For some variable x and some valuations v_1 and v_2 , we define the permutation π_{x, v_1, v_2} on D as the transposition which swaps $v_1(x)$ and $v_2(x)$, and the renaming Π_{v_1, v_2} as the collection of these permutations, variable per variable:

$$\pi_{x, v_1, v_2}(a) = \begin{cases} v_1(x) & \text{if } a = v_2(x) \\ v_2(x) & \text{if } a = v_1(x) \\ a & \text{otherwise} \end{cases}; \quad (28)$$

$$\Pi_{v_1, v_2} = (\pi_{x, v_1, v_2})_{x \in X}. \quad (29)$$

Note that these definitions are symmetric in v_1 and v_2 . Moreover note that $\Pi_{v_1, v_2}(v_1) = v_2$ and $\Pi_{v_1, v_2}(v_2) = v_1$.

Consider a formula $F \in \mathcal{F}$ and a solution τ of F . We denote by $\chi_F(\tau)$ the set of valuations σ assigning each variable x one of the values in the set $A_F(\tau, x)$:

$$\chi_F(\tau) = \prod_{x \in X} A_F(\tau, x). \quad (30)$$

When τ is a solution of F , we denote by $\mathcal{C}(\tau, F)$ the set of all renamings of (τ, F) ranging in $\chi_F(\tau)$:

$$\mathcal{C}(\tau, F) = \{(\sigma, \Pi_{\sigma, \tau}(F))\}_{\sigma \in \chi_F(\tau)}. \quad (31)$$

Lemma 35. If τ is a solution of F and $\sigma \in \chi_F(\tau)$, then σ is a solution of $G = \Pi_{\sigma, \tau}(F)$ and for any variable $x \in X$, $A_G(\sigma, x) = A_F(\tau, x)$.

Proof. Since $G = \Pi_{\sigma, \tau}(F)$, $\sigma = \Pi_{\sigma, \tau}(\tau)$ and τ is a solution of F , by Fact 34 we know that σ is a solution of G . Moreover by Fact 34, for any variable x , $A_G(\sigma, x) = \pi_{x, \sigma, \tau}(A_F(\tau, x))$. By the definition of χ_F , $\sigma(x) \in A_F(\tau, x)$. Since $\pi_{x, \sigma, \tau}$ swaps two values $\tau(x)$ and $\sigma(x)$ that are both elements of $A_F(\tau, x)$, $\pi_{x, \sigma, \tau}(A_F(\tau, x)) = A_F(\tau, x)$, hence $A_G(\sigma, x) = A_F(\tau, x)$. \square

Lemma 36. The set $\{C(\tau, F)\}_{\substack{F \in \mathcal{F} \\ \tau \in \mathcal{M}_{<}(F)}}$ is a partition of \mathcal{C} .

Proof. If $(\sigma, G) \in C(\tau, F)$, then by Lemma 35, σ is a solution of G . Moreover, by closure of \mathcal{F} under renaming, $G \in \mathcal{F}$. Thus $C(\tau, F) \subseteq \mathcal{C}$. Now it is sufficient to prove that $\forall (\sigma, G) \in \mathcal{C}$ there exists a unique (τ, F) where $F \in \mathcal{F}$, τ is a minimal solution of F and $(\sigma, G) \in C(\tau, F)$.

- Existence of (τ, F) : for any x , let $\tau(x)$ be the minimal value in $A_G(\sigma, x)$ according to the order $<_x$ underlying $<$; by construction, $\tau \in \chi_G(\sigma)$. Consider the renaming $\Pi_{\sigma, \tau}$ and let $F = \Pi_{\sigma, \tau}(G)$. By Lemma 35, τ is a solution of F and for any variable x , $A_F(\tau, x) = A_G(\sigma, x)$. Since for any $x \in X$, $\tau(x)$ is the minimal value in $A_F(\tau, x)$, τ is minimal for the orientation $<$. Moreover for any x , $\sigma(x) \in A_F(\tau, x)$, thus $\sigma \in \chi_F(\tau)$; and since $G = \Pi_{\sigma, \tau}(F)$, we have that $(\sigma, G) \in C(\tau, F)$.
- Uniqueness of (τ, F) : let (τ', F') be such that $C(\tau', F') \ni (\sigma, G)$, i.e. $\sigma \in \chi_{F'}(\tau')$ and $G = \Pi_{\sigma, \tau'}(F')$; then by Lemma 35, for any variable x , $A_G(\sigma, x) = A_{F'}(\tau', x)$. By minimality of τ' , $\tau'(x)$ must be the minimum of $A_G(\sigma, x)$ for each variable x . \square

Lemma 37. Suppose that the weight w_F is obtained from a unitary and instance independent generator ω . Let (τ, F) be an element of \mathcal{C} ; then $\sum_{(\sigma, G) \in C(\tau, F)} W_G(\sigma) = 1$.

Proof. First note that by Lemma 35, for any $(\sigma, G) \in C(\tau, F)$, we have $A_G(\sigma, x) = A_F(\tau, x)$. Thus:

$$\begin{aligned}
 \sum_{(\sigma, G) \in C(\tau, F)} W_G(\sigma) &= \sum_{(\sigma, G) \in C(\tau, F)} \prod_{x \in X} w_G(\sigma, x) \\
 &= \sum_{(\sigma, G) \in C(\tau, F)} \prod_{x \in X} \omega(x, \sigma(x), A_G(\sigma, x)) \text{ since } \omega \text{ is instance independent} \\
 &= \sum_{\sigma \in \chi_F(\tau)} \prod_{x \in X} \omega(x, \sigma(x), A_F(\tau, x)) \text{ since } A_G(\sigma, x) = A_F(\tau, x) \\
 &= \prod_{x \in X} \sum_{\sigma(x) \in A_F(\tau, x)} \omega(x, \sigma(x), A_F(\tau, x)) \\
 &= \prod_{x \in X} 1 \text{ since } \omega \text{ is unitary} \\
 &= 1. \quad \square
 \end{aligned}$$

Theorem 38. Let \mathcal{F} be a set of CSP instances which is closed under renaming. Let w_F be a weighting system built from a unitary and instance independent weight generator ω . Let $<$ be an instance independent orientation. Then it holds that $\sum_{F \in \mathcal{F}} |\mathcal{M}_{<}(F)| = \gamma(\mathcal{F})$.

Proof. It is a mere combination of Lemmas 36 and 37:

$$\begin{aligned}
 \gamma(\mathcal{F}) &= \sum_{F \in \mathcal{F}} \gamma(F) \\
 &= \sum_{F \in \mathcal{F}} \sum_{\sigma \in \mathcal{S}(F)} W_F(\sigma) \\
 &= \sum_{(\sigma, F) \in \mathcal{C}} W_F(\sigma) \\
 &= \sum_{\substack{F \in \mathcal{F} \\ \tau \in \mathcal{M}_{<}(F)}} \sum_{(\sigma, G) \in C(\tau, F)} W_G(\sigma) \text{ by Lemma 36} \\
 &= \sum_{\substack{F \in \mathcal{F} \\ \tau \in \mathcal{M}_{<}(F)}} 1 \text{ by Lemma 37} \\
 &= \sum_{F \in \mathcal{F}} |\mathcal{M}_{<}(F)|. \quad \square
 \end{aligned}$$

Closure under renaming involves symmetry, so it is not surprising that on average all weightings on the one hand and all orderings on the other hand should be equivalent. What is more surprising though, is the fact that weightings and orderings are equivalent. This is noteworthy because weights are simpler to handle in calculations; for example Boufkhad and Hugel [8] use weights to derive more compact and tractable formulas. It turns out that the best weighting they end up with is in fact an ordering. This is not a consequence of the preceding observations because they make use of a non-uniform weighting, which is out of the scope of this paper.

8. Conclusion and perspectives

Through our Weight Conservation Theorem we gave sufficient conditions to have a correct weighting on solutions of CSPs. We were able to apply it to two different weightings: the first one, which is very general, was built from seeds and dispatchers; the second one was specifically designed to improve on Maneva et al.'s weighting. Thanks to this new weighting scheme, we obtained an improvement on the upper bound on the existence of non-trivial cores obtained by Maneva et al. to 4.419.

We also showed an equivalence between weighting and ordering over a set closed under renaming when they are instance independent. On the contrary, when weighting and ordering may depend on instances, we showed that given an homogeneous weighting it is possible to find an ordering which is not worse, but what happens for heterogeneous weightings? Is it always possible to find for a given weighting a corresponding ordering?

Other perspectives include: is it possible to define a correct non-uniform weighting? Indeed, in [8], a non uniform way of weighting and orienting is defined. The non uniformness in this context means that the weightings depend not only on instances and values but may vary from a solution to another. Other questions may be addressed: How to generalize trivaluations to general CSPs? How to extend our weighting when considering neighbors of neighbors, or more generally neighbors at bounded distance?

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Appendix. Proof of Theorem 22

A.1. First moment of cores

We strongly advise the reader to read Maneva and Sinclair [17] before reading our calculations, since we reuse all of the notation and arguments from there. So we only highlight the similarities and the differences.

The calculation of Maneva and Sinclair [17] works in two steps:

1. compute an upper bound function on the first moment f of covers and discard the range of variables where $f < 0$ (because if there are no covers, then there are no cores either);
2. compute an upper bound function of the first moment $f + h$ of cores and maximize it on the remaining domain of the variables.

In particular they introduce some variables s, t, u ; to these we add v :

- s : the size of the cover or core i.e. the number of variables in a controlled self-constrained set (they get a weight of 1); we denote them by symbols x_i where $i \in SC = \{1, \dots, s\}$;
- v : number of invertible variables (they get a weight of $\frac{1-\rho}{2}$); we denote them by symbols x_i where $i \in I = \{s+1, \dots, s+t+v\}$;
- $u - v$: proportion of starrable but non-invertible variables (they get a weight of $1 - \rho$); we denote them by symbols x_i where $i \in SNI = \{s+t+v+1, \dots, s+u\}$;
- $t - u$: number of non-starrable variables not in the previous self-constrained set (they get a weight of 1); we denote them by symbols x_i where $i \in NS = \{s+u+1, \dots, s+t\}$;
- $n - s - t$: number of variables at value $*$ (they get a weight of ρ); we denote them by symbols x_i where $i \in S = \{s+1, \dots, s+t-u\}$.

p is the probability for a clause of type 3 to be included in the Poisson model. In Table A.1 we sum up all possible types of clauses in order to count them. We assume by symmetry that we have an assignment with values in $\{0, *\}$ only (no 1's). Note that clauses of type 1 and 2 are the same as in [17] and are used in the expression of f rather than in the expression of h .

As Maneva and Sinclair [17] did, we first define the following quantity:

$$Q = (1-p)^{\binom{t}{3} + s\binom{t}{2} + 2(n-s-t)\left(\binom{t}{2} + st\right) + u\left(\binom{t}{2} + st\right) + 2v(n-s-t)(s+t)} \\ \times \left(1 - (1-p)^{\binom{t}{2} + st}\right)^{t-u} \left(1 - (1-p)^{2(n-s-t)(s+t)}\right)^{u-v}.$$

Table A.1

Clauses and their sizes.

| Types | Clause | Sets of subscripts | Size | Status |
|-------|-------------------------------------|---|--|-----------------------------------|
| 1 | $x_i \vee x_j \vee x_k$ | $i, j, k \in SC$ | $\binom{s}{3}$ | Forbidden |
| | $x_i \vee x_j \vee \bar{x}_k$ | $i, j \in SC, k \in V \setminus SC$ | $2(n-s)\binom{s}{2}$ | Forbidden |
| | $x_i \vee x_j \vee \bar{x}_k$ | | | |
| 2 | $x_i \vee x_j \vee \bar{x}_k$ | $i, j, k \in SC$ | $\binom{s}{2}$ for each x_k | At least one for each $k \in SC$ |
| 3 | $x_i \vee x_j \vee x_k$ | $i \in SC \cup NS \cup I \cup SNI,$ $j, k \in NS \cup I \cup SNI$ | $\binom{t}{3} + s\binom{t}{2}$ | Forbidden |
| | $x_i \vee x_j \vee x_k$ | $i \in SC \cup NS \cup I \cup SNI,$ $j \in NS \cup I \cup SNI$ $k \in S$ | $2(n-s-t)\left(\binom{t}{2} + st\right)$ | Forbidden |
| | $x_i \vee x_j \vee \bar{x}_k$ | | | |
| | $x_i \vee x_j \vee \bar{x}_k$ | $i \in SC \cup NS \cup I \cup SNI,$ $j \in NS \cup I \cup SNI$ $k \in I \cup SNI$ | $u\left(\binom{t}{2} + st\right)$ | Forbidden |
| | $x_i \vee \bar{x}_j \vee x_k$ | $i \in SC \cup NS \cup I \cup SNI$ $j \in I$ $k \in S$ | $2(n-s-t)v(s+t)$ | Forbidden |
| | $x_i \vee \bar{x}_j \vee \bar{x}_k$ | | | |
| | $x_i \vee x_j \vee \bar{x}_k$ | $i \in SC \cup NS \cup I \cup SNI$ $j \in NS \cup I \cup SNI$ $k \in NS$ | $\binom{t}{2} + st$ for each x_k | At least one for each $k \in NS$ |
| | $x_i \vee \bar{x}_j \vee x_k$ | $i \in SC \cup NS \cup I \cup SNI$ $j \in SNI$ $k \in S$ | $2(n-s-t)(s+t)$ for each x_j | At least one for each $j \in SNI$ |
| | $x_i \vee \bar{x}_j \vee \bar{x}_k$ | | | |
| | $x_i \vee \bar{x}_j \vee \bar{x}_k$ | | | |

Now with Q we can write down the first moment of cores having $n - s - t$ variables at value $*$:

$$\begin{aligned}
 EZ_t &= \rho^{n-s-t} \binom{n-s}{t} 2^t \sum_{u=0}^t (1-\rho)^u \binom{t}{u} \sum_{v=0}^u 2^{-v} \binom{u}{v} Q \\
 &= \rho^{n-s-t} \binom{n-s}{t} 2^t (1-p)^{\binom{t}{3} + s\binom{t}{2} + 2(n-s-t)\left(\binom{t}{2} + st\right)} \\
 &\quad \times \sum_{u=0}^t (1-\rho)^u \binom{t}{u} (1-p)^{u\left(\binom{t}{2} + st\right)} \left(1 - (1-p)^{\binom{t}{2} + st}\right)^{t-u} \\
 &\quad \times \sum_{v=0}^u 2^{-v} \binom{u}{v} (1-p)^{2v(n-s-t)(s+t)} \left(1 - (1-p)^{2(n-s-t)(s+t)}\right)^{u-v} \\
 &= \rho^{n-s-t} \binom{n-s}{t} 2^t (1-p)^{\binom{t}{3} + s\binom{t}{2} + 2(n-s-t)\left(\binom{t}{2} + st\right)} \\
 &\quad \times \sum_{u=0}^t (1-\rho)^u \binom{t}{u} (1-p)^{u\left(\binom{t}{2} + st\right)} \left(1 - (1-p)^{\binom{t}{2} + st}\right)^{t-u} \left(1 - \frac{(1-p)^{2(n-s-t)(s+t)}}{2}\right)^u \\
 &= \rho^{n-s-t} \binom{n-s}{t} 2^t (1-p)^{\binom{t}{3} + s\binom{t}{2} + 2(n-s-t)\left(\binom{t}{2} + st\right)} \\
 &\quad \times \left((1-\rho)(1-p)^{\binom{t}{2} + st} \left(1 - \frac{(1-p)^{2(n-s-t)(s+t)}}{2}\right) + 1 - (1-p)^{\binom{t}{2} + st} \right)^t \\
 &= \rho^{n-s-t} \binom{n-s}{t} 2^t (1-p)^{\binom{t}{3} + s\binom{t}{2} + 2(n-s-t)\left(\binom{t}{2} + st\right)} \\
 &\quad \times \left(1 - (1-p)^{\binom{t}{2} + st} \left(\rho + \frac{(1-\rho)}{2} (1-p)^{2(n-s-t)(s+t)} \right) \right)^t.
 \end{aligned}$$

$s = \lfloor an \rfloor$, $t = bn$ and $p = \frac{3\alpha(1-d)}{n^2(4-a^2(3-a))} + o\left(\frac{1}{n^2}\right)$, so:

$$h = \lim_{n \rightarrow +\infty} \frac{\ln EZ_t}{n} = \ln \left(\frac{(1-a)^{1-a} 2^b \rho^{1-a-b}}{b^b (1-a-b)^{1-a-b}} \right) - \frac{\alpha(1-d)b(b(6-5b-3a) + 12(1-a-b)a)}{2(4-a^2(3-a))}$$

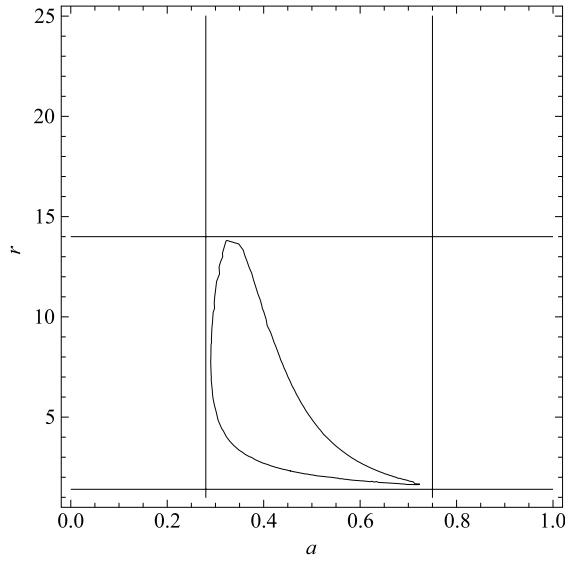


Fig. A.3. $f(4.419, a, r) > -0.0001$ inside the contour line, so only when $a \in [0.28, 0.75]$ and $r \in [1.4, 14]$.

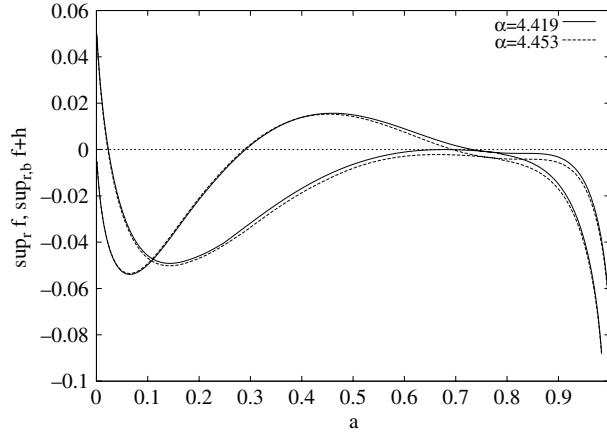


Fig. A.4. f and $f + h$ at $\alpha = 4.453$ and at $\alpha = 4.419$ (with our weights).

$$+ b \ln \left(1 - e^{-A} \left(\rho + \frac{1 - \rho}{2} e^{-B} \right) \right)$$

$$\text{where } A = \frac{3\alpha(1-d)b(b+2a)}{2(4-a^2(3-a))} \text{ and } B = \frac{6\alpha(1-d)(1-a-b)(a+b)}{(4-a^2(3-a))}.$$

A.2. Maximization

Just like Maneva and Sinclair [17], we want to show that for any $\alpha \in [4.419, 4.453]$, $a \in \left[\frac{1}{4.453e^2}, 1 \right]$ and $r > 1$, when $\rho(a) = 0.3758a + 0.7067$, either $f(\alpha, a, r) < 0$ or for any $b \in [0, 1 - a]$, $f(\alpha, a, r) + h(\alpha, a, r, \rho(a), b) < 0$. Note that our function f is the same as in [17]; only h differs.

As in [17], if $r < 1.2$ then $\frac{\partial f}{\partial r} > 0$, and if $r > 670$ then $\frac{\partial f}{\partial r} < 0$, so we are left with the region $a \in \left[\frac{1}{4.453e^2}, 0.999 \right]$, $r \in [1.2670]$. The points where $f(4.419, a, r) > -0.0001$ are depicted in Fig. A.3. This corresponds to the domain where we must check that $f + h$ is negative. We get bounds slightly different from [17]: $a \in [0.28, 0.75]$, $r \in [1.4, 14]$, $\alpha \in [4.419, 4.453]$ and $b \in [0, 1 - a]$.

Fig. A.4 gives the shapes of our f and $f + h$ at $\alpha = 4.419$ and at $\alpha = 4.453$.

The domain of the variables is a finite product of segments, all functions involved are smooth, except at the boundary points where $b = 1 - a$ (but this is due to the asymptotic equivalent we used for the binomial coefficient). So in order to maximize $f + h$, following again Maneva and Sinclair [17], we performed a sweep over this domain with a step of 0.001 on

all variables (and a step of 10^{-5} in the vicinity of the maximum). In the end we checked our result using the FindMaximum function of Mathematica. The maximum of $f + h$ is -0.0000277225 , and the values of the variables at this point are $\alpha = 4.419$, $a = 0.678206$, $b = 0.0299196$ and $r = 1.79833$.

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